Quantum Morphisms

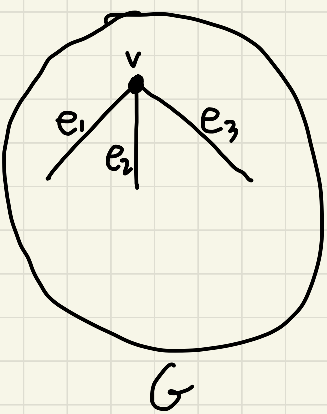
Lecture 11

Last Week

1) G a connected graph w/ incidence mtx M, and $b \in \mathbb{Z}_2^{V(G)}$ with odd weight. Then

- $Mx = b$ has NO solution
- $Mx = b$ has a (finite dim.) quantum solution if & only if G is NOT planar.

We view the edges of G as our variables and its vertices as our equations:



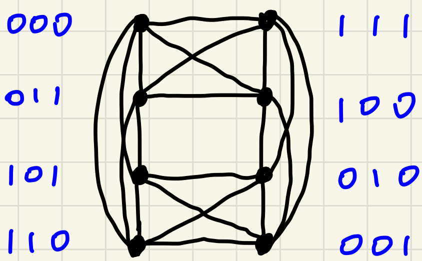
Equation corresponding to vertex v:

$$x_{e_1} + x_{e_2} + x_{e_3} = b_v$$

2) Given a BLS $Mx=b$, we construct the graph $G(M,b)$

- Vertices: satisfying assignments $f: S_\ell \to \mathbb{Z}_2$ to the equations in $Mx=b$.
- Adjacent if they disagree.

Example: $x_1+x_2+x_3=0$, $x_1+x_4+x_6=1$



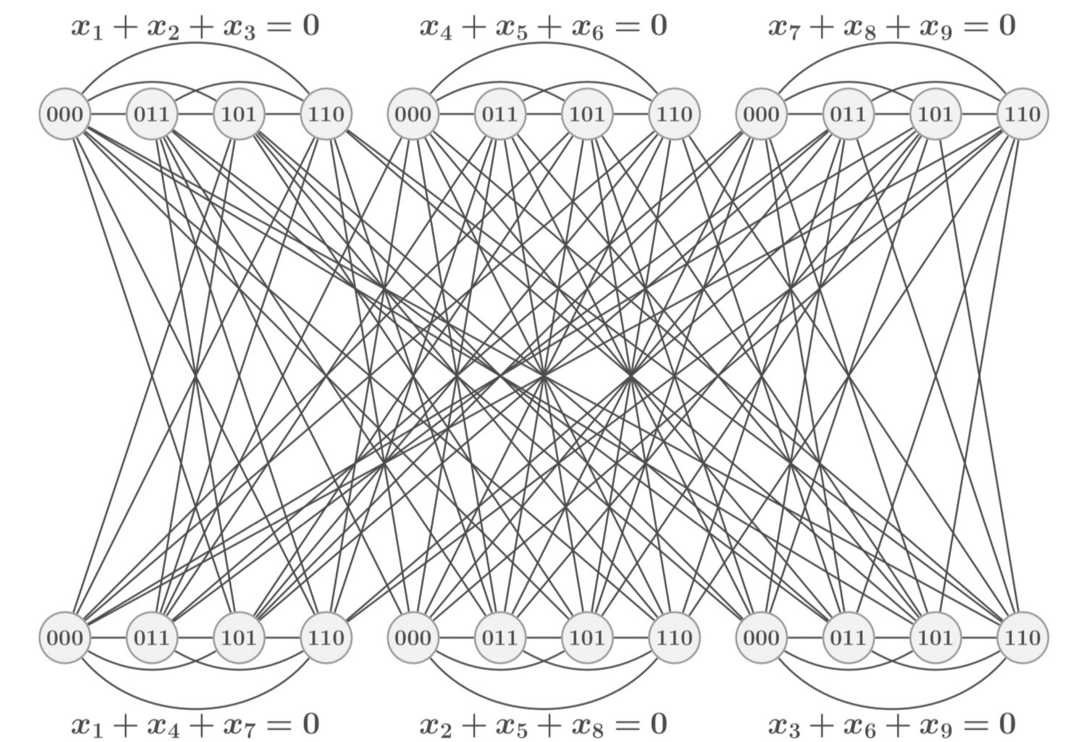
3) Theorem:

$Mx=b$ has a solution $\Leftrightarrow G(M,b) \cong G(M,0)$

$Mx=b$ has a quantum solution $\Leftrightarrow G(M,b) \cong_{qc} G(M,0)$

$Mx=b$ has a fin. dim. quantum solution $\Leftrightarrow G(M,b) \cong_q G(M,0)$

Corollary: Let G be a connected non-planar graph w/ incidence mtx M and $b \in \mathbb{Z}_2^{V(G)}$ w/ odd weight. Then $G(M,b) \cong_q G(M,0)$, but $G(M,b) \not\cong G(M,0)$.


$x_1 + x_2 + x_3 = 0$
$x_4 + x_5 + x_6 = 0$
$x_7 + x_8 + x_9 = 0$
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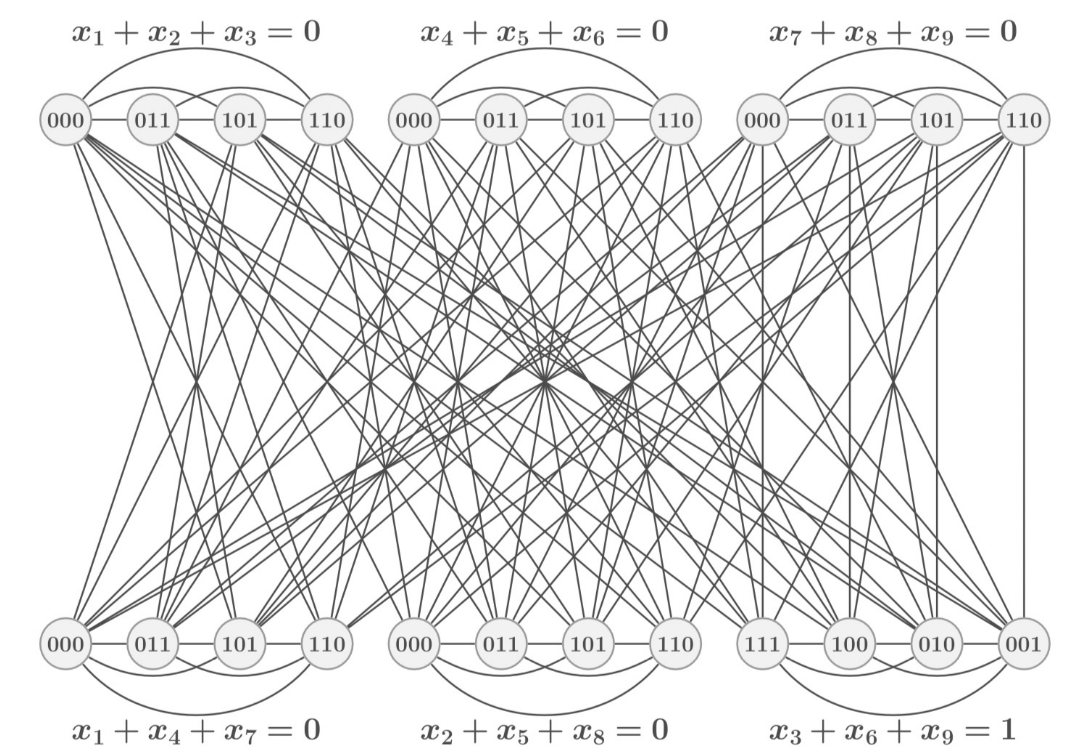
$x_1 + x_4 + x_7 = 0$
$x_2 + x_5 + x_8 = 0$
$x_3 + x_6 + x_9 = 0$


$x_1 + x_2 + x_3 = 0$
$x_4 + x_5 + x_6 = 0$
$x_7 + x_8 + x_9 = 0$
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$x_1 + x_4 + x_7 = 0$
$x_2 + x_5 + x_8 = 0$
$x_3 + x_6 + x_9 = 1$

Reminder

Theorem: $G \cong_{qc} H$ if + only if there exists a C^*-algebra $\mathcal{A}$ that admits a tracial state, and there is a quantum permutation matrix (QPM) $\mathcal{P} \in M_n(\mathcal{A})$ s.t.

$$A_G \mathcal{P} = \mathcal{P} A_H. \quad (*)$$

(g,h)-entry of $(*)$: $\displaystyle\sum_{g' \sim g} p_{g'h} = \sum_{h' \sim h} p_{gh'}$

For a QPM, $(*) \Leftrightarrow p_{gh} p_{g'h'} = 0$ if $rel(g,g') \neq rel(h,h')$.

Recall: $\mathcal{P} = (p_{ij}) \in M_n(\mathcal{A})$ is a QPM if

- $p_{ij} = p_{ij}^2 = p_{ij}^* \quad \forall\ i,j \in [n]$
- $\sum_k p_{ik} = 1 = \sum_\ell p_{\ell j} \quad \forall\ i,j \in [n]$

Quantum Groups

A compact matrix quantum group (CMQG) $\mathbb{G}$ is a pair $(C(\mathbb{G}), \mathcal{U})$ where $C(\mathbb{G})$ is a unital C^*-algebra which is generated by the entries of the matrix $\mathcal{U} = (u_{ij}) \in M_n(C(\mathbb{G}))$. Moreover, the $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ given by $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ must exist, and $\mathcal{U}$ and its transpose $\mathcal{U}^T$ must be invertible.

Motivating Example:

Let $\mathbb{G}$ be a subgroup of $GL(n, \mathbb{C})$. We use $C(\mathbb{G})$ to denote the algebra of continuous functions from $\mathbb{G}$ to $\mathbb{C}$ under pointwise multiplication. Let $u_{ij} : \mathbb{G} \to \mathbb{C}$ denote the function that maps an element of $\mathbb{G}$ to its ij-entry. Then $(C(\mathbb{G}), \mathcal{U})$ where $\mathcal{U} = (u_{ij}) \in M_n(C(\mathbb{G}))$ is a CMQG.

Conversely, any CMQG $\mathbb{G} = (C(\mathbb{G}), \mathcal{U})$ where $C(\mathbb{G})$ is commutative is isomorphic to a CMQG of this form.

In the noncommutative case $C(\mathbb{G})$ is still often referred to as "the algebra of functions on $\mathbb{G}$".

Automorphism Group of a Graph

$Aut(G) = \{ P \in \mathbb{C}^{V(G) \times V(G)} :$ P is a perm. mtx. & $A_G P = P A_G \}$.

Let $u_{ij} : Aut(G) \to \mathbb{C}$ be defined as in the example above.

Claim: The u_{ij} generate $C(Aut(G))$. $\cong \mathbb{C}^{Aut(G)}$

Proof: Let $P \in Aut(G)$ & define $\pi : V(G) \to V(G)$ s.t.

$$P_{ij} = \begin{cases} 1 & \text{if } j = \pi(i) \\ 0 & \text{o.w.} \end{cases}$$

Then $u_\pi = \prod_{i \in V(G)} u_{i\pi(i)}$ is the characteristic function of P:

$$u_\pi(P') = \prod_{i \in V(G)} P'_{i\pi(i)} = \begin{cases} 1 & \text{if } P' = P \\ 0 & \text{o.w.} \end{cases}$$

Claim: $\mathcal{U}=(u_{ij})$ is a QPM. Moreover $A_G\mathcal{U}=\mathcal{U}A_G$.

Proof: $u_{ij}(P)\in\{0,1\} \Rightarrow u_{ij}(P)=u_{ij}^{*}(P)^2=u_{ij}^{*}(P)^{*}\ \forall P\in Aut(G)$

$$\Rightarrow u_{ij}=u_{ij}^2=u_{ij}^*$$

$\sum_k u_{ik}(P)=1\ \ \forall P\in Aut(G) \Rightarrow \sum_k u_{ik}=1$

Similarly $\sum_\ell u_{\ell j}=1$.

Proof of $A_G\mathcal{U}=\mathcal{U}A_G$ left as exercise.

Defining $C(Aut(G))$ abstractly

We want a quantum analog of $Aut(G)$, i.e. a noncommutative version of $C(Aut(G))$.

$a,b \quad a^2=b^2=id \quad ab=ba$

Define $\mathcal{A}(G)$ to be the universal C^*-algebra generated by elements p_{ij} for $i,j\in V(G)$ satisfying the relations:

$$\left.\begin{array}{l} 1)\ p_{ij}=p_{ij}^2=p_{ij}^* \quad \forall i,j\in V(G) \\ 2)\ \sum_k p_{ik}=1=\sum_\ell p_{\ell j} \quad \forall i,j\in V(G) \end{array}\right\}\ \mathcal{P}=(p_{ij}) \text{ is a QPM}$$

3) $A_G \mathcal{P} = \mathcal{P} A_G$

4) the p_{ij} all commute

Universal C^*-algebra construction is analogous to defining groups using generators and relations.

This means that if $\mathcal{A}'$ is a C^*-algebra generated by some elements $p'_{ij} \in \mathcal{A}'$ for $i,j \in V(G)$ and the p'_{ij} satisfy relations (1)–(4), then there is a surjective $*$-homomorphism $\phi: \mathcal{A}(G) \to \mathcal{A}'$ s.t. $\phi(p_{ij}) = p'_{ij}$.

Proposition: There is a $*$-isomorphism $\phi: \mathcal{A}(G) \to C(Aut(G))$ s.t. $\phi(p_{ij}) = u_{ij}$.

Proof: There is a surjective $*$-homomorphism ϕ by universality of $\mathcal{A}(G)$. Prove this is injective.

The Quantum Automorphism Group

(Banica)

Define $C(Qut(G))$ to be the universal C^*-algebra generated by elements u_{ij} for $i,j \in V(G)$ satisfying the relations:

$$\left.\begin{array}{l} 1)\ u_{ij} = u_{ij}^2 = u_{ij}^* \quad \forall\, i,j \in V(G) \\ 2)\ \sum_k u_{ik} = 1 = \sum_\ell u_{\ell j} \quad \forall\, i,j \in V(G) \end{array}\right\} \ \mathcal{U} = (u_{ij}) \text{ is a QPM}$$

3) $A_G \mathcal{U} = \mathcal{U} A_G$

Then $Qut(G) = (C(Qut(G)), \mathcal{U})$ is a CMQG called the quantum automorphism group of G.

The matrix $\mathcal{U}$ is called the fundamental representation.

Special Case: $G = \overline{K_n}$ (empty graph on n vertices)

Then $Qut(G) = S_n^+$ – the quantum symmetric group.

$K_1 : \mathcal{U} = (1) \qquad K_2 : \mathcal{U} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$

Remark: Sometimes $C(Qut(G))$ is commutative, e.g. $G = K_n$ for $n \leq 3$, or $G = C_n$ for $n \neq 4$, or if G is the Petersen graph (Schmidt)

In this case we write $Qut(G) = Aut(G)$ and say G has no quantum symmetry.

Remark: $Qut(G) = Qut(\overline{G})$ since any QPM commutes with $I + J$ and $A_{\overline{G}} = J - I - A_G$.

Properties of $Qut(G)$

Comultiplication: $\Delta : C(Qut(G)) \to C(Qut(G)) \otimes C(Qut(G))$

$\Delta(ab) = \Delta(a)\Delta(b)$
$\Delta(a^*) = \Delta(a)^*$

$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ is a $*$-hom

Antipode: $S : C(Qut(G)) \to C(Qut(G))^{op}$ $\quad S(ab) = S(b)S(a)$

$S(u_{ij}) = u_{ji}$ is a $*$-hom

Counit: $\varepsilon : C(Qut(G)) \to \mathbb{C}$

$\varepsilon(u_{ij}) = \delta_{ij}$ is a $*$-hom

Haar state: $h : C(Qut(G)) \to \mathbb{C}$ satisfying

$(h \otimes id) \circ \Delta = (id \otimes h) \circ \Delta = h$

For $Qut(G)$, h is tracial.

Intertwiners

$\mathcal{U}^{\otimes k}$ - $V(G)^k \times V(G)^k$ matrix with

$$(\mathcal{U}^{\otimes k})_{i_1 \cdots i_k, j_1 \cdots j_k} = u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_k j_k}$$

$$\mathcal{U}^{\otimes 0} = (1)$$

An (ℓ,k)-intertwiner of $Qut(G)$ is a matrix $T \in \mathbb{C}^{V(G)^\ell \times V(G)^k}$ s.t. $\mathcal{U}^{\otimes \ell} T = T \mathcal{U}^{\otimes k}$.

$A_G \mathcal{U} = \mathcal{U} A_G$

Examples: A_G is a $(1,1)$-intertwiner by definition.

$$(M^{1,2})_{i,jj'} = \begin{cases} 1 & \text{if } i=j=j' \\ 0 & \text{o.w.} \end{cases} \quad \text{i.e. } M^{1,2} e_i \otimes e_j = \delta_{ij} e_i$$

is a $(1,2)$-intertwiner. Exercise.

$M^{1,0} = \sum_{i \in V(G)} e_i$ (the all 1's vector) is a $(1,0)$-intertwiner.

$$\mathcal{U} M^{1,0} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = M^{1,0} \mathcal{U}^{\otimes 0}$$

$C_q^G(\ell,k) :=$ set of (ℓ,k)-intertwiners of $Qut(G)$.

$C_q^G := \bigcup_{\ell,k} C_q(\ell,k)$

Proposition: C_q^G is a tensor category with duals, i.e.

$$\mathcal{U}^{\otimes \ell} \otimes \mathcal{U}^{\otimes r} = \mathcal{U}^{\otimes \ell + r}$$

1) $C_q^G(\ell,k)$ is a vector space
2) $T \in C_q^G(\ell,k)$, $T' \in C_q^G(r,s) \Rightarrow T \otimes T' \in C_q^G(\ell+r, k+s)$
3) $T \in C_q^G(\ell,k)$, $T' \in C_q^G(k,r) \Rightarrow TT' \in C_q^G(\ell,r)$
4) $T \in C_q^G(\ell,k) \Rightarrow T^* \in C_q^G(k,\ell)$
5) $I \in C_q^G(1,1)$
6) $\psi = \sum_{i \in V(G)} e_i \otimes e_i \in C_q^G(2,0)$

Tannaka-Krein duality (Woronowicz)

The correspondence between a CMQG $\mathbb{G} \subseteq O_n^+$ and its intertwiners is a one-to-one correspondence between such $\mathbb{G}$ and tensor categories with duals contained in $\bigcup_{\ell,k} \mathbb{C}^{n^\ell \times n^k}$.

Remark: If $\mathbb{G} \subseteq O_n^+$ is a CMQG, then $C(\mathbb{G})$ is commutative if & only if S (defined as $S e_i \otimes e_j = e_j \otimes e_i$) is an interwiner of $\mathbb{G}$. Exercise.

Theorem (Chassaniol):

$$C_q^G = \langle M^{1,2}, M^{1,0}, A_G \rangle_{+,\circ,\otimes,*}$$

$$C^G = \langle M^{1,2}, M^{1,0}, A_G, S \rangle_{+,\circ,\otimes,*}$$

Intertwiners of $Aut(G)$.

Classical Case

Let $\mathcal{U}$ be the fundamental representation of $Aut(G)$. Then

$$\mathcal{U}^{\otimes \ell} T = T \mathcal{U}^{\otimes k} \Leftrightarrow P^{\otimes \ell} T = T P^{\otimes k} \quad \forall\, P \in Aut(G).$$

Thus $T \in C^G$ if & only if T is constant on the orbits of the action of $Aut(G)$ on $V(G)^\ell \times V(G)^k$, i.e. $T_{i_1 \cdots i_\ell, j_1 \cdots j_k} = T_{i_1' \cdots i_\ell', j_1' \cdots j_k'}$ if $\exists\, \pi \in Aut(G)$ s.t. $\pi(i_r) = i_r'$ & $\pi(j_s) = j_s'$ $\forall\, r, s$.

$C^G(\ell, k)$ = span of characteristic matrices of orbits of $Aut(G)$ on $V(G)^\ell \times V(G)^k$.

In the quantum case, we can also define a notion of orbits of $Qut(G)$ on $V(G)^{\ell} \times V(G)^{k}$ if $\ell + k \leq 2$.